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Innovative Applications of O.R.

Spatial Lanchester models

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ABSTRACT

Lanchester equations have been widely used to model combat for many years, nevertheless, one of their most important limitations has been their failure to model the spatial dimension of the problems. Despite the fact that some efforts have been made in order to overcome this drawback, mainly through the use of Reaction–Diffusion equations, there is not yet a consistently clear theoretical framework linking Lanchester equations with these physical systems, apart from similarity. In this paper, a spatial modeling of Lanchester equations is conceptualized on the basis of explicit movement dynamics and balance of forces, ensuring stability and theoretical consistency with the original model. This formulation allows a better understanding and interpretation of the problem, thus improving the current treatment, modeling and comprehension of warfare applications. Finally, as a numerical illustration, a new spatial model of responsive movement is developed, confirming that location influences the results of modeling attrition conflict between two opposite forces.

1. Introduction

Lanchester equations (LEs) were introduced by F.W. Lanchester as a set of linear differential equations that describe an attrition conflict between two opposite forces concentrated on a spot, as Kimball (1950) reports. Since then, LEs have been widely used to model and theorize about combat attrition for many years. See for example Chen and Chu (1991), Kaup et al. (2005) and Hung et al. (2005) for some recent contributions. This success can be explained mainly because of the simplicity of Lanchester equations (LEs), and the fact that they are very intuitive and hence easy to apply. Additionally, at present there exist several research lines that are using LEs to analyze very distinct problems, such as: Adams and Mesterson-Gibbons (2003) in behavioral ecology, Lacasta et al. (2008) in infectology, Kimball (1957), Erickson (1997) in marketing and Hirshleifer (1991) in economics, which have maintained the interest in the Lanchester approach.

The Lanchester model makes strong simplifying assumptions that have proven to be important shortcomings in terms of real-battle outcomes forecasting. Nowadays, most modern warfare simulations are stochastic, heterogeneous and complex and in general terms give better predictions than the traditional LEs. Thus, important efforts have been made in order to generalize Lanchester original formulation and improve its performance, e.g. stochastic and heterogeneous models have been introduced, for details see Grubbs and Shuford (1973), Taylor (1974, 1983), Taylor and Brown (1983), Chen (2002), Roberts and Conolly (1992). Despite this, little attention has been paid on one of its main limitations: the fact that LEs fail to model the characteristic spatial dimension of most attrition problems.

Location matters in the evolution and state of a struggle, whether some entity is fighting a war, defining its marketing campaign or its vaccinations programs. The capacity of modeling different spatial settings in a consistent and stable manner is crucial at the time of deciding an army strategy, since it allows the modeler to take into account local battles and disaggregated allocation of resources, but at the same to keep in mind the global strategy.

A publication by Protopopescu et al. (1989) was the first work that tried to model combat in a spatial setting via Lanchester equations, including one-dimensional spatial effect. This first attempt was followed by Cosner et al. (1990) that used a parabolic system with nonlinear interactions. Fields (1993) explicitly modeled the displacement of forces in a 2D model, and most recently Spradlin and Spradlin (2007) and Keane (2009) have tried to use and expand Protopopescu's work to two dimensions.

All these attempts have concentrated in the use of partial differential equations to model combat, as Reaction–Diffusion equations systems. However, to this date and apart from their similarities, there is not a consistent and clear theoretical framework linking Lanchester equations with this physical system. As a consequence of this, interpretation of the diffusion and strategic behavior of the forces in a spatial setting could have not been properly addressed in the literature. Particularly, in these formulations there is not an explicit account for displacements, reinforcements and deaths of the engaged forces, and some stability issues are not discussed at all.

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In this paper, a spatial modeling of Lanchester equations is conceptualized on the basis of an explicit balance of forces and developed in order to account not only for the time dynamics of the problem, but also for locations, movements and concentrations of the struggling forces. The resulting formulation ensures stability and theoretical consistency with the original model, allowing for a better understanding and interpretation of the spatial simulation. Besides, in order to complement the general model, the dynamics of the spatial combat is explicitly defined for some cases: troops movements, terrain modeling, responsive movement, perception, predator–prey behavior, and distance combat. It is expected, that this new taxonomy could certainly improve not only the warfare applications, but also the new research projects inspired by LEs, including stochastic behavior of both the result of the combat, as mentioned by Grubbs and Shuford (1973), Hellman (1996), Gass (1997), and the displacement of the forces, as stated by Fields (1993).

Additionally, as a numerical illustration, a new model of responsive movement is developed. In other words, the model includes the movement of the forces according to the balance of gradients of both own and enemy‘s troops and terrain effects. The spatio-temporal simulations confirm the fact that location influences the results of modeling attrition conflict between two opposite forces.

The rest of this paper is organized as follows. In the next section, a general formulation of the new model and its equations are presented. A continuity equation is developed accounting for displacements, generations or reinforcements and deaths of the engaged forces, that is to say: a balance of forces. In section three, the dynamics of the spatial modeling is defined. Thus, some different combat situations that could yield different movements of the forces and hence different diffusion and velocity characteristics are presented. In section four, a comparison of the model to previous works is done. In the fifth section a particular case of spatial combat is modeled: responsive movement. In section six a numerical example is developed, showing how location and concentration of forces matter in combat results. Finally, the findings are summarized, and directions for future research are discussed.

2. The general model

The first assumption needed to model the spatial Lanchester problem is to use a spatial coordinates system for the forces that will engage in combat, typically two: the Red and Blue armies. Thus, without loss of generality, the surface density, or number of soldiers per area unit, of the Blue forces will be represented by $B(x,y,t)$ and that of the Red forces will be represented by $R(x,y,t)$. An element of each force will have an instantaneous velocity given by $\vec{v}_\theta(x,y,t)$ where θ can be replaced either by B or R. So the surface

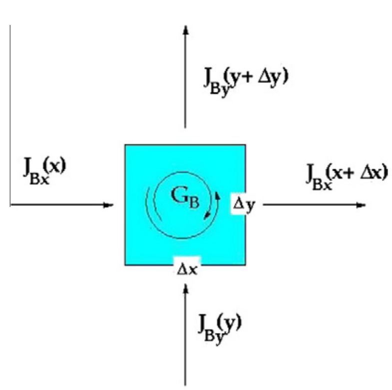

Fig. 1. Divergence of the Blue forces through an element of surface.

densities of current, or number of soldiers traveling parallel to the velocity per transversal distance unit, of the Blue and Red forces are represented by $\vec{J}_\theta(x,y,t) = \theta(x,y,t)\vec{v}_\theta(x,y,t)$.

It should be highlighted that both surface densities B and R must be non-negatively valued functions if soldiers are the elements of the forces.

Both forces will meet in a region of area $\Delta x \Delta y$. For this purpose Fig. 1 depicts the continuity evolution of the Blue forces.

Now the temporal variation of the flow density, or the number of forces that actually cross the transversal distance per time unit, of the B forces coming into the region of area $\Delta x \Delta y$ plus the internal generation or reinforcement G_B is $\Delta \Phi_B$. Hence the instantaneous time variation of the Blue forces in the region should be expressed as:

$$\Delta \Phi_B = -[J_{By}(y+\Delta y) - J_{By}(y)]\Delta x - [J_{Bx}(x+\Delta x) - J_{Bx}(x)]\Delta y + G_B \Delta x \Delta y. \tag{1}$$

For the R forces, there is an analog expression that results from changing the subscripts to R.

As a result of the combat, that includes decay, spontaneous generation, regeneration or reinforcements, destruction and self-destruction, a nonlinear net result of each force is obtained. The respective density of the generic Blue or Red forces θ resulting from the struggle, is described by:

$$\Delta \Phi_\theta = \frac{\partial \theta}{\partial t} \Delta x \Delta y. \tag{2}$$

By imposing the continuity relation and taking the limit when both Δx and Δy vanish, and by replacing (2) into (1), a general expression is obtained:

$$G_\theta - \vec{\nabla} \cdot \vec{J}_\theta = G_\theta - \frac{\partial J_{\theta x}}{\partial x} - \frac{\partial J_{\theta y}}{\partial y} = \frac{\partial \theta}{\partial t}. \tag{3}$$

It remains clear that $\vec{J}_\theta = R \cdot \vec{v}_\theta$, where $\vec{v}_\theta$ is the instantaneous velocity of each part of the respective moving force. Without losing generality the same equations could be extended to a volumetric combat, adding easily the z coordinate. For the purpose of this document that discussion is left out.

The internal densities can be expressed considering the profile of the engaging forces through the Lanchester expressions:

$$G_B = g_B(x,y,t) - \sum_{i=0}^{\infty} \left(\sum_{j=0}^{\infty} \alpha_{Bij} R^i B^j \right), \tag{4}$$

where the α coefficients have the same interpretation as in the original LEs, being always real valued and generally space–time dependent. There is an analog expression for G_R.

Combining Lanchester Eqs. (4) and (3) and the constitutive relations for $\vec{J}_B$ a new expression arises:

$$G_B - \vec{\nabla} \cdot (B\vec{v}_B) = -\vec{\nabla} \cdot \vec{J}_B + g_B(x,y,t) - \sum_{i=0}^{\infty} \left(\sum_{j=0}^{\infty} \alpha_{Bij} R^i B^j \right) = \frac{\partial B}{\partial t} \tag{5}$$

and applying the same procedure, a twin expression results.

Eq. (5) represents the general approach to spatially modeling Lanchester equations, and they can be generically written as:

$$-\vec{\nabla} \cdot (\theta \vec{v}_\theta) = \frac{\partial \theta}{\partial t} - G_\theta, \tag{6}$$

where θ can be either B or R.

This equation is general and should apply to any conflict engaging two forces. The left hand side term reflects the behavior of attraction/repulsion of the forces while the term G_θ accounts for the birth and death governing the evolution of the combat. The analysis found in the next paragraphs illustrates better this matter, specifically some particular assumptions are discussed.

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It is useful to recall that the total number of remaining forces in the battlefield for any time is described by:

$$\theta_T = \int\int_S \theta(x,y,t)dS. \quad (7)$$

It is important to remark that the formulation presented above has four novel and important features that are not properly addressed in the literature on spatial attrition modeling. Firstly, this model is explicitly and consistently derived from the Lanchester's original formulation and hence it is not constructed from its similarity to some physical systems. Secondly, this model is not a specific case, because it presents a general approach, allowing the inclusion of different kinds of behavior for the forces, including diffusive attitudes, attractive or repulsive, among others. Thus, arbitrary and unjustified assumptions are avoided. Thirdly, the correct and explicit balance of forces is modeled, guaranteeing that no soldiers or forces can arbitrarily appear or disappear. In fact it allows to account for the forces at any time. Usually Fick's law, as indicated by Dekker (1959), Smith (2004) can be applied with the balance of forces in a continuous setting if stability and consistency have to be achieved. Finally, a 2D formulation is derived, which can be very helpful for didactic and visual purposes.

3. Defining the spatial dynamics of attrition modeling

In order to bound the solution to the general problem already described, the dynamics of the spatial modeling must be defined.

Thus, it is possible to identify some combat situations that could rise from the attitudes of the forces. We discuss each situation in turn, starting with the more basic.

3.1. Troop directed movement without other effects

Since most movements on the battlefield are directed towards a specific location or target, a basic movement of the forces will be to follow a given path. It is assumed that the forces are not responsive to any other information or force but they will just follow their orders unless they fight, fact that can change the average velocity due to the local killing rate. In absence of other troops the movement is represented by:

$$\vec{J}_B = B\vec{v}_{0B}. \quad (8)$$

Under these assumptions, $\vec{v}_{0B}$ is only dependent on attrition or on endogenous decisions, and it is not necessarily constant.

3.2. Troop directed movement influenced by terrain

Terrain can have different effects on each of the struggling forces whether they are homogeneous or not. In general, the troops and soldiers velocities will critically depend on the terrain conditions. Whether or not the surface is plane or sloped will certainly affect the troops' movements. Usually, the troops could move in the direction of the negative gradient of the terrain topology defined by a function: $\psi(x,y)$. In other words:

$$\vec{J}_{BT} = -\sigma\vec{\nabla}\psi(x,y). \quad (9)$$

In general, σ is a real valued proportionality constant or a terrain dependent real valued function, and is also known as diffusion coefficient.

In this case, a constant component will account for friction, but with no relevance in the global analysis.

If no attrition is happening, then:

$$\vec{J}_B = B\vec{v}_B = B\vec{v}_{0B} - \sigma\vec{\nabla}\psi(x,y), \quad (10)$$

where $\psi(x,y)$ represents the terrain so the path should be influenced by the surface once the commanders have decided to move with $\vec{v}_{0B}(x,y,t)$, for example: aiming north from the south in a straight line. Under this rule of movement the terrain will act as a perturbation on the commander's order.

3.3. Responsive movement and perception

Usually in a war situation, it is expected that the motion of one of the forces, even if combats were discarded, might influence the motion of the other one. This type of movement is called responsive movement. Assuming that the density of current of forces will be related to the balance of both forces, two simple possibilities should be born in mind as main drivers of the forces (not the only ones): linear behavior of the velocities and linear behavior of the densities of current of forces.

The way a force reacts during the struggle may vary widely depending on the type of forces involved. Moreover, perception of the strength can be very different if it regards own forces or opposite ones. Lack of intelligence is an extreme way in the behavior of a force, but high use of the knowledge of the own and opposing motion of each side could be close to the other extreme, especially if the reaction is nonlinear.

Four parameters can be introduced here which account for a combined effect of the perception and ability of each force. The first subscript indicates the observer and the second subscript indicates the subject of the observation, e.g.: h_{BR} is the result of such combined effect of strength of the Red forces perceived by the Blue forces. In order to separate both effects, it is useful to state the actual strength of each force: k_B and k_R, so four pure parameters (u_{ij}) shape just perception, and they are defined as follows:

$$\begin{bmatrix} h_{BB} & h_{BR} \\ h_{RB} & h_{RR} \end{bmatrix} = \begin{bmatrix} k_B u_{BB} & k_R u_{BR} \\ k_B u_{RB} & k_R u_{RR} \end{bmatrix}. \quad (11)$$

Each u_{ij} should take positive real values, reflecting the quality of intelligence of each army. When heterogeneous forces are fighting, each of the u_{ij} terms have to be extended to an array represented by one sub-matrix.

As a rule of thumb, for the simplest case $h_{ij} = k_j u_{ij}$.

This way, for the B force, u_{BB} indicates how confidently the B forces located on the (x,y) spot know themselves. On the other hand, u_{BR} express how good is the knowledge that the B forces have of the strength of the R forces. The same criteria applies for the perceptions of the Red forces.

3.3.1. Direct relation between the velocity and a weighted difference of the gradients of the two forces

The velocity of the forces will move towards or away from its enemies depending on the balance of gradients. This time other terms already discussed are included, friction and commanded velocity $\vec{v}_{0B}$, and for the sake of simplicity the terrain effects have been left out:

$$\frac{\vec{J}_B}{B} = \vec{v}_B = -\vec{s}_B - w_B(h_{BB}\vec{\nabla}B - h_{BR}\vec{\nabla}R) + \vec{v}_{0B}. \quad (12)$$

The term $\vec{s}_B$ is a real valued function that represents the velocity component of friction, while the parameter w_B is a real valued proportionality constant.

By replacing (12) into (3) the densities of current no longer appear in the equations, leaving the problem with a resemblance to the classic Poisson equation in terms of the B and R forces densities:

$$\frac{\partial B}{\partial t} - G_B = -\vec{v}_{0B}\cdot\vec{\nabla}B + \vec{s}_B\cdot\vec{\nabla}B + w_B h_{BB}(|\vec{\nabla}B|^2 + B\nabla^2 B) - w_B h_{BR}(\vec{\nabla}B\cdot\vec{\nabla}R + B\nabla^2 R). \quad (13)$$

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In this case the problem is intrinsically nonlinear in B and R. This equation can be trivially modified for the dynamics of the Red forces.

3.3.2. Direct relation between the density of current and a weighted difference of the gradients of the two forces

The group of forces located at some position will move towards or away from the other forces according to the perceived strength of the opponent.

$$\vec{J}_B = -\vec{f}_B - p_B(h_{BB}\vec{\nabla}B - h_{BR}\vec{\nabla}R) + B\vec{v}_{0B}, \tag{14}$$

where p_B and p_R are proportionality constants and $\vec{f}_B$ and $\vec{f}_R$ are friction terms, as described early.

By replacing (14) into (3) and proceeding in the same way for the other forces, the densities of current no longer appear in the equations, leaving the problem with a resemblance to the classic Poisson equation in terms of the B and R force densities:

$$p_B\left(h_{BB}\nabla^2 B - h_{BR}\nabla^2 R\right) - \vec{v}_{0B} \cdot \vec{\nabla}B = \frac{\partial B}{\partial t} - G_B \tag{15}$$

and also an analog equation obtained by swapping the B and R subscripts.

Due to the right side of this equation, as Eq. (4) expresses, in general this is still a nonlinear problem.

3.4. Nonlocal movements and attrition

Two effects arise from nonlocal struggle, movement and destruction. These consequences are analyzed here.

3.4.1. Struggle-driven movement: Predator–prey behavior

It is expected that an army will move towards or away the enemies depending on its own balance of forces. In simple terms, fighting units are expected to direct their fire at a single specific opposite unit that they consider they can destroy. The notion of accessibility, whether they can win or not, obviously will depend upon their expectations of superiority. Thus, a tentative form of "intelligence" under a perfect information assumption that can evaluate a unit superiority against a determined target can be expressed as follows:

$$\int_{t_0}^{t_0+T}\int\int_A (Bk_B - Rk_R)\,dA\,dt \geqslant 0. \tag{16}$$

Fig. 2 shows the incremental analysis needed to formulate a displacement policy for the B forces.

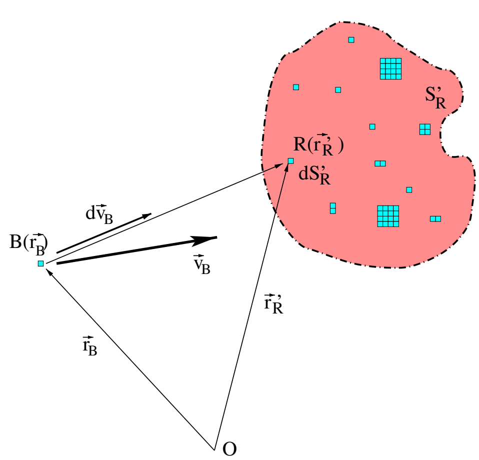

Fig. 2. Displacement Interaction over the *B* forces.

If Eq. (16) holds true, then B is going to move towards R, if not, B is going to escape in the opposite direction or evaluate a different target. On the other hand, the velocity of $\vec{v}_B$, once the direction is set, is going to depend upon the distance of the two forces.

Assuming that an element of the Blue forces moves in the orientation where it perceives extreme weakness or robustness, a spatial function should define the attitude of that element according to the relative distance while moving on the line that joins the positions of the antagonistic elements of the forces. As perceptions are involved, the effect of integration over the S'_R domain can drive the B forces through a twisting path, away from the one obtained when information is perfect. Red forces can experience the same winding in their movements.

$$\vec{v}_B = \int\int_{S_R} \mu_B\left(\|\vec{r}'_R - \vec{r}_B\|, h_{BR}R(\vec{r}'_R) - h_{BB}B(\vec{r}_B), z_B\right)\frac{\vec{r}'_R - \vec{r}_B}{|\vec{r}'_R - \vec{r}_B|}\,dS'_R. \tag{17}$$

The great importance of the introduction of this kind of velocity is that it exists even if no elements of the force are present at some spot, as it happens with any potential function. Also, it should be highlighted that this interaction would induce a force alignment for the combat or the escape, quite long before.

3.4.2. Nonlocal attrition

If attrition takes place at a significant distance, for example due to the use of artillery, the killing rate over each unit of the $B(\vec{r}_B)$ force would be needed. If it is assumed that a nonlocal portion of the R forces inflict losses from that remote location, then by using the notation given in Eq. (3), the relative contribution to the decay of each elementary individual of the B forces can be calculated from the effect of the remote portion of the enemy forces $R(\vec{r}'_R)$. That value is:

$$d\left(\frac{G_B}{B}\right)\bigg|_{remote} = d\left(\frac{1}{B}\frac{dB}{dt}\right)\bigg|_{remote}. \tag{18}$$

In this model R forces fire according to their ammunition stock $a_R(\vec{r}'_R)$, distance to the B forces $\|\vec{r}'_R - \vec{r}_B\|$, relative targeting on the B forces $z_B(\vec{r}_B)$ (no information about the ammunition distribution of the enemy) and perceived dis-balance of the struggling forces $\delta(\vec{r}_B, \vec{r}'_R) = h_{RR}R(\vec{r}'_R) - h_{RB}B(\vec{r}_B)$. The relative targeting on the B forces is assumed to be conditioned by the perception observed by the R forces over the B forces, that can also be a function of the spatial coordinates.

$$z_B(\vec{r}_B) = \frac{h_{RB}B(\vec{r}_B)\int\int_{S'_B} dS'_B}{\int\int_{S'_B} h_{RB}B(\vec{r}'_B)\,dS'_B}. \tag{19}$$

And the relative rate of firing of the R forces located at $\vec{r}'_R$ should be η_R, which can be expressed as a function of the four already mentioned variables.

Hence, the absolute contribution to the spatial Lanchester equations is:

$$G_B|_{remote} = \frac{dB}{dt}\bigg|_{remote}, \tag{20}$$

or

$$G_B|_{remote} = -B(\vec{r}_B)\int\int_{S'_R} R(\vec{r}'_R)\eta_R\left(\|\vec{r}'_R - \vec{r}_B\|, z_B(\vec{r}_B), a_R(\vec{r}'_R), \delta(\vec{r}_B, \vec{r}'_R)\right)dS'_R, \tag{21}$$

4. Comparison to other formulations

In an early work, Protopopescu et al. (1989) model the spatial behavior of the forces, using only one spatial dimension:

$$\frac{\partial B}{\partial t} = \vec{v}_{0B} \cdot \vec{\nabla}B + \vec{\nabla} \cdot \left(D_1\vec{\nabla}B\right) + I_B. \tag{22}$$

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This model can be interpreted with the forces simultaneously starting to fight and trying to stay around or chasing their neighboring adversaries. The authors explicitly left the explanation of the parameters for future work. By recurring to the model shown in Eq. (6), it is easy to reverse-engineer the formulations of the authors already mentioned:

$$\vec{v}_B = -\vec{v}_0 - \frac{1}{B}\left(\vec{f}_0 + D_1\vec{\nabla}B\right). \tag{23}$$

Moreover, that formulation implicitly assumes forces that have diffusive velocity components due to each of the engaging forces, so the forces run away from their high concentration of forces to their own low concentrations, disregarding the concentrations of the enemies.

As mentioned at the introduction, given the lack of a conceptual framework properly linking LEs with the Reaction–Diffusion equations, the correct interpretation of this model become a very complex task.

The model expressed by other authors such as Cosner et al. (1990) that follows up Protopopescu, does not interpret the terms in the equations, e.g.: velocities, focusing on the mathematical solution of a general equation. The model presented by Fields (1993) implicitly includes anisotropy in the diffusion, but later on when using a scalar as the unique constant of diffusion, a negative constant vector is added to the velocity field of the struggling forces, mentioning only that the components of that constant vector are called velocity constants. For these authors the expression for the time evolution of the Blue forces is:

$$\frac{\partial B}{\partial t} = \vec{v}_{0B}\cdot\vec{\nabla}B + \vec{\nabla}\cdot(D_1\vec{\nabla}(B+R)) + I_B. \tag{24}$$

In all the cases cited here, the constant velocity is not an initial velocity because at that time the term ∇^2 is not zero everywhere, unless the distribution of the force (forces) is (are) not constant. But Fields (1993), assumes that the velocity of the Blue forces becomes:

$$\vec{v}_B = -\vec{v}_0 - \frac{1}{B}(\vec{f}_0 + D_P\vec{\nabla}(B+R)). \tag{25}$$

An innovative feature of this formulation is that the velocity depends on the sum of the forces, which could be considered the first responsive movement model. However, this model implicitly assumes forces that have diffusive velocity components due to each of the engaging forces, so the forces run away from the crowds no matter if they are friends or foes.

In an attempted expansion to 2D, Spradlin and Spradlin (2007) write:

$$\frac{\partial B}{\partial t} = -\vec{v}_{0B}\cdot\vec{\nabla}B - I_B, \tag{26}$$

where I_B represents the attrition rate, and the velocity is a constant, so movement is not present at all. Also, no attraction or repulsion is described:

$$\vec{v}_B = \vec{v}_0 - \frac{1}{B}\vec{f}_0. \tag{27}$$

A variant of the other models appears when Keane (2009) includes a term called the time derivative of the forces that is assumed to be composed of three parts, one due to diffusion, another one to the interaction and the third one as a result of the velocities. In this formulation, again the diffusion term is not regarded as a velocity component as it should be:

$$\frac{\partial B}{\partial t} = I_B + \vec{\nabla}\cdot(D_B(B)\vec{\nabla}B) + \vec{\nabla}\cdot\{B(\vec{C}_B B + A_a(K_a * B) + A_R B(K_R * B))\} + I_B. \tag{28}$$

The actual velocity that emerges from that formulation is:

$$\vec{v}_B = -\vec{C}_B B - A_a(K_a * B) - A_R B(K_R * B) - \frac{1}{B}(\vec{f}_0 + D_B(B)\vec{\nabla}(B)), \tag{29}$$

where A_a and A_R are constants, and K_a and K_R must be vectorial operators, not described by the authors.

It is perhaps due to the problems of explicit definitions pointed out above in the original Protopopescu's seminal work, that only a few papers have followed up this line of research. Indeed, with limited theoretical development and shallow interpretation, it could be difficult to generate useful spatial applications. Thus, most of the work on spatial attrition models previously discussed could benefit from the explicit and consistent general framework of analysis introduced here, allowing a more clear understanding and interpretation of the spatial modeling. It is expected that from this clarification, further work on spatial LE will be developed and not only for warfare applications.

As a conclusion of the analysis already presented, this new spatial specification of LEs can be seen as a general formulation, bringing a taxonomy of models and explicit definitions for spatial Lanchester Models.

5. A new model of responsive local movement

Even though any of the formulations presented above could be analytically and numerically implemented, this section develops an original responsive movement dynamics, which highlights the combat intelligence rather than the troop movement. In any case, it is important to remark that these two dynamics (troop directed and responsive/intelligent movement) are not competitive and can be seen as complementary. For purposes of explanation, the analytical and numerical example here developed is thought to show combat dynamics in the battlefield, where emphasis is given to the combat intelligence.

Specifically, a responsive movement of the soldiers will be modeled. Thus, the densities of currents of the forces will move towards or away from the enemies depending on the balance of their gradients. In other words, each force will evaluate dynamically its strength against the opposite forces, and it will move accordingly.

Note that if friction is neglected and $h_{BB}/h_{BR} = h_{RB}/h_{RR}$, then both forces will have the same instantaneous direction at the same point of the surface, but with different signs. This condition is equivalent to having the determinant of the associated linear system equal to zero.

5.1. An application on the time side

The situation to be analyzed assumes no spontaneous generation (no reinforcements) and only linear dependencies, whereas the terms from Eq. (4),

$$\begin{aligned} &\alpha_{B10} = E_R(= k_R) \quad \alpha_{B01} = M_B, \\ &\alpha_{R10} = E_B(= k_B) \quad \alpha_{R01} = M_R \end{aligned} \tag{30}$$

are used in order to compare with the example of Bach et al. (1962), and thus results in:

$$[\Theta]\begin{bmatrix} B \\ R \end{bmatrix} - \frac{\partial}{\partial t}\begin{bmatrix} B \\ R \end{bmatrix} = 0, \tag{31}$$

where the $[\Theta]$ matrix operator is:

$$[\Theta] = \begin{bmatrix} p_B h_{BB}\nabla^2 - M_B - (\vec{v}_{0B}\cdot\vec{\nabla}) & -p_B h_{BR}\nabla^2 - E_R \\ -p_R h_{RB}\nabla^2 - E_B & p_R h_{RR}\nabla^2 - M_R - (\vec{v}_{0R}\cdot\vec{\nabla}) \end{bmatrix}. \tag{32}$$

5.2. Numerical solution for a 2D uniform grid

As B and R must be non-negatively valued for each point of the space–time domain, the solution to this problem can not be treated as in the Dirichlet (zero-order boundary conditions) or Neumann (first-order boundary conditions) problem, because the non-negativity can be regarded as a time-dependent border condition. Bearing in mind this statement, a numerical approach suits this problem better, where the formulation given by (31) drives to a time-stepping formulation for the spatial profile of the B and R scalar fields. The procedure should provide a way to adjust the time step so as to limit the maximum deviation of negative values and then to reset acceptable deviations to zero. The time stepping can be faced with a stable Crank–Nicolson (C–N) method, leaving the spatial problem to other methods. Here, Finite Differences (FD) are used as a first approach.

For further details on the FD formulation see the proposal by Gonzalez and Villena (2009).

6. Numerical illustration

A very simple numerical example is developed in order to show the potential of the model described above. A new formulation is presented for simulating a combat under Lanchester square law rules while the forces are being attracted or repelled accordingly to the perception of their local relative strength, like in (31) but when $\vec{v}_{0B} = \vec{v}_{0R} = 0$. Clearly this new formulation has not been modeled using the already mentioned classical or spatial attrition approaches, and it consists of a simple combat with responsive movement.

The situation is described by two forces, one is an occupying force uniformly spread over the domain, the same as the rebel local army that uses the same area, but gets visible at time $t = 0$.

$M_B = +3$; net self decay of the Blue forces
$E_R = +1$; death rate of the Blue forces caused by the Red forces
$M_R = +1$; net self decay of the Red forces
$E_B = +8$; death rate of the Red forces caused by the Blue forces

The occupying army (Blue forces) has a high killing rate (eight times as effective as the Red forces), but also a higher decaying rate due to the lack of local support, having to face a self decay rate three times the value of the local army (Red forces). The forces will engage on a one-to-one base, as in the Lanchester square law, due to the hard chance to fight on open space. They simultaneously start fighting while trying to stay around or chasing their neighboring adversaries. It is assumed that the battlefield is so broad that the boundary conditions allow to have zero forces on the perimeter of the domain.

Under these conditions, it is important to recognize when one of the forces reach a 5% of its initial value, figure that marks the end of the battle. Also it is important to see how the forces spread during the combat. Because the forces are in the right place, some dispersion is expected as a result of the normal diffusion that generates the presence of the enemies. This spreading, enemy-driven as already shown, is important because it might explain a different outcome of one battle.

Once the battle starts, the aim of each army is total annihilation of the opposite force. In this particular case, the individuals move by diffusion only according to the local balance of forces, assuming the behavior explained in Section 5, in Subsection 5.1 and $\vec{v}_{0B} = \vec{v}_{0R} = 0$. Using the same values as in the benchmark, but accepting one on one interaction among the fighting forces, this example shows the spatial behavior during the conflict. This time, in order to avoid abrupt changes in the initial state of the system, the following functions are used for the distribution of the forces:

$$\theta(x, y, t = 0) = \gamma_\theta \cdot sech\left(\frac{m_\theta}{c_\theta^2}\left[(x - x_0)^2 + (y - y_0)^2\right]\right), \tag{33}$$

where θ can be B or R, $\gamma_B = 2000$, $\gamma_R = 6000$, c_θ has been set as the length of eighteen cell sides and $m_\theta = 4$ is a dimensionless modulating constant.

Stability is granted for this situation and is reported in the work of Gonzalez and Villena (2009).

Using the same initial concentration of each force for two simulations, where the scenery is altered by shifting the location of the engaging forces in zero and fourteen cells apart, a first result shows that the combat lasts less if the forces are placed initially over the same location. On the other extreme, the combat lasts more when there is a significant shift between the center of each force. Even though this spatial formulation assumes a very basic form of intelligence, this example aims to focus on the space–time evolution of the problem rather than in the effectiveness of the chosen diffusion definition.

Thus, evolution of each force and comparison between different initial locations show how the distribution of the spatial forces is altered, as well as the resulting time for a practical annihilation of the weaker force. The square grid for the simulations uses 53 points on each side and the total space span represented on each side is $52 \cdot 10^{-3}$ in arbitrary units, so each cell is a square of side 10^{-3}. The time variable uses a step of $0.1 \cdot 10^{-3}$ in arbitrary units. Proportionality constants p_B and p_R are taken as 10^{-5} in arbitrary units. And as the forces have no intelligence, all the four u parameters are unitary.

The arbitrary units taken, can account for a plausible situation where the physical units are given together with the equivalences in Table 1:

From Figs. 3 and 4 it is clear that the final distribution of the Blue Forces depends on the shift of the centroid of the forces. The effect also happens with the Red Forces, condition that can be seen in Figs. 5 and 6. As expected, Figs. 3 and 5 are symmetric in relation to the diagonals, and to the center lines. All the surfaces are plotted using *UC* and *cell side* units.

It must be remarked that this model accounts for the change in combat duration regarding the classical Lanchester equations, situation that is explained by the diffusive behavior of this kind of forces.

It can be pointed out that even though the second combat lasts longer, the number of casualties of the triumphant rebels (Red force) is significantly reduced. When physical units are used, the total amount of rebels is initially 23,757 soldiers while the sieging army has 7919 soldiers. Evolution times are compared for 95%

Table 1
Equivalences and units.

Concept	Arbitrary unit	Physical unit
Time	*UT*	531.91 [minutes]
Distance	*UL*	10^5 metres
Cell side	10^{-3} *UL*	100 metres
Time step	10^{-4} *UT*	3.1915 seconds
Soldiers	*US*	10^4 soldier
Concentration of Forces (B or R)	$UC = US \cdot UL^{-2}$	10^{-6} soldier metre^{-2}
Velocity ($\vec{v}$) or Speed (v)	$UV = UL \cdot UT^{-1}$	188 metres minutes^{-1}
Density of current ($\vec{J}$)	$UJ = US \cdot UL^{-1} \cdot UT^{-1}$	$1.88 \cdot 10^{-3}$ soldier metres^{-1} minutes^{-1}
p_B	10^{-5} UL^2	10^5 metres^2
p_R	10^{-5} UL^2	10^5 m^2
M_B	3 UT^{-1}	0.00564 minutes^{-1}
E_R	1 UT^{-1}	0.00188 minutes^{-1}
M_R	1 UT^{-1}	0.00188 minutes^{-1}
E_B	8 UT^{-1}	0.01504 minutes^{-1}

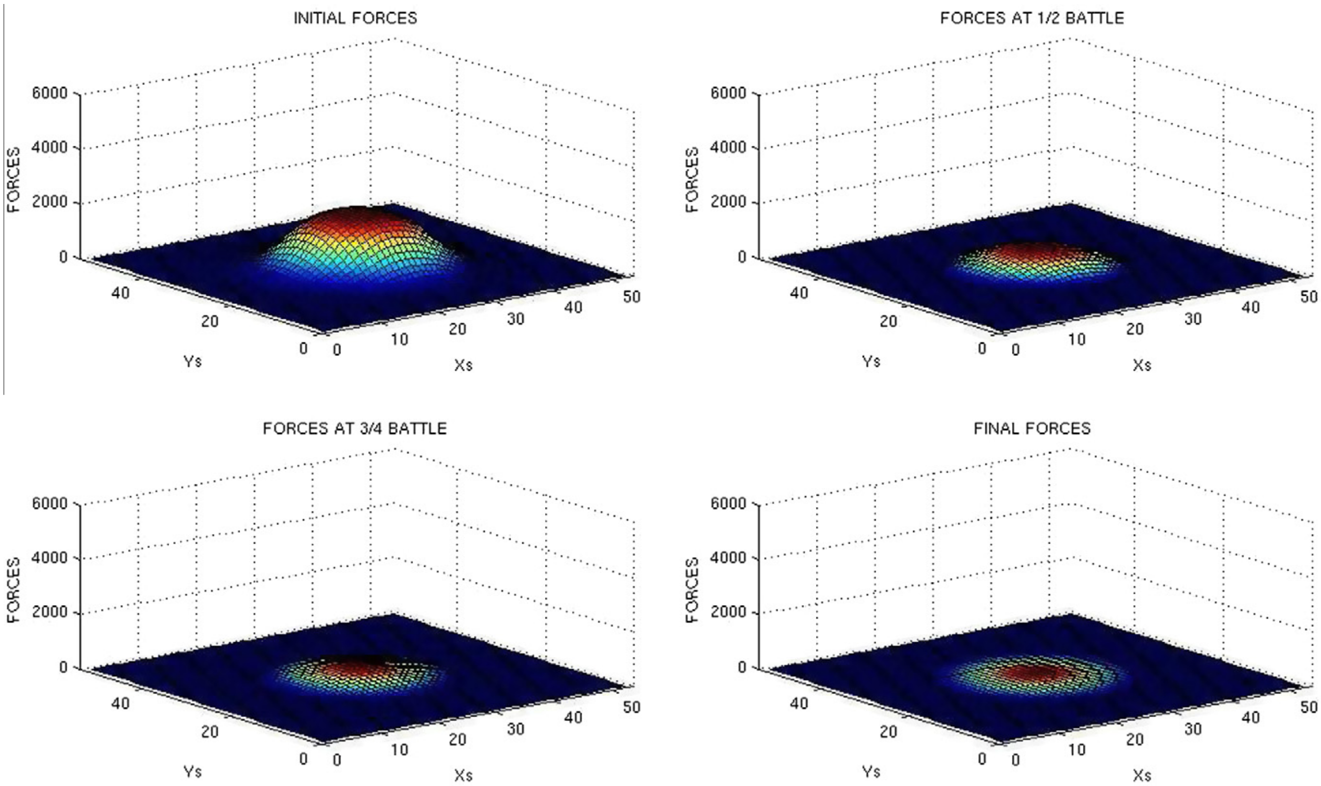

Fig. 3. Evolution of Blue forces for centered combat. $B(x,y)$ vs. spatial coordinates.

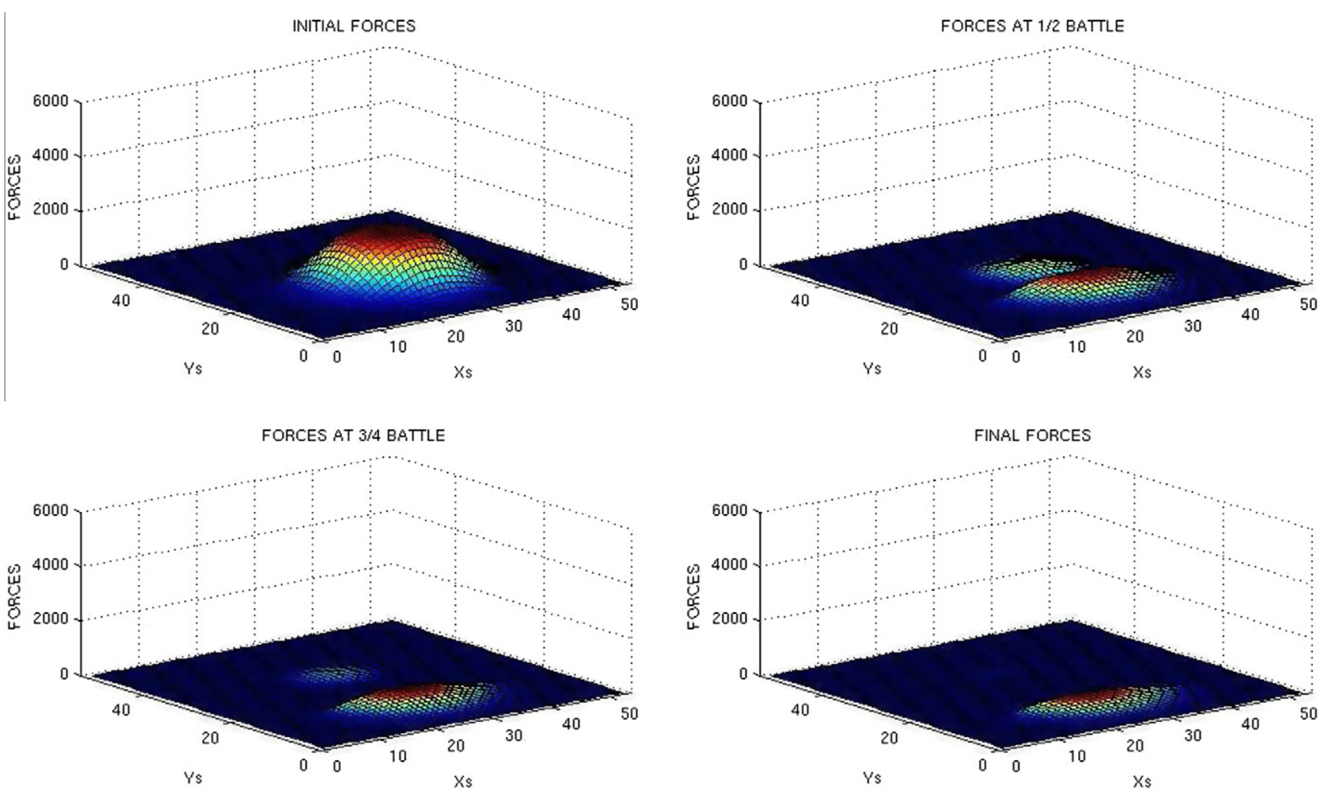

Fig. 4. Evolution of Blue forces for shifting. $B(x,y)$ vs. spatial coordinates.

annihilation of one of the forces: the results obtained show some unexpected behavior, where ${}^{0}t_{95} = 318.72$ minutes but for off-center combat ${}^{14}t_{95} = 341.06$ minutes. A plausible explanation can be found in the unbalanced values of E_R, E_B, M_R and M_B that resemble the example analyzed by Bach et al. (1962). Interestingly, the overall results change from the original LEs, and more information

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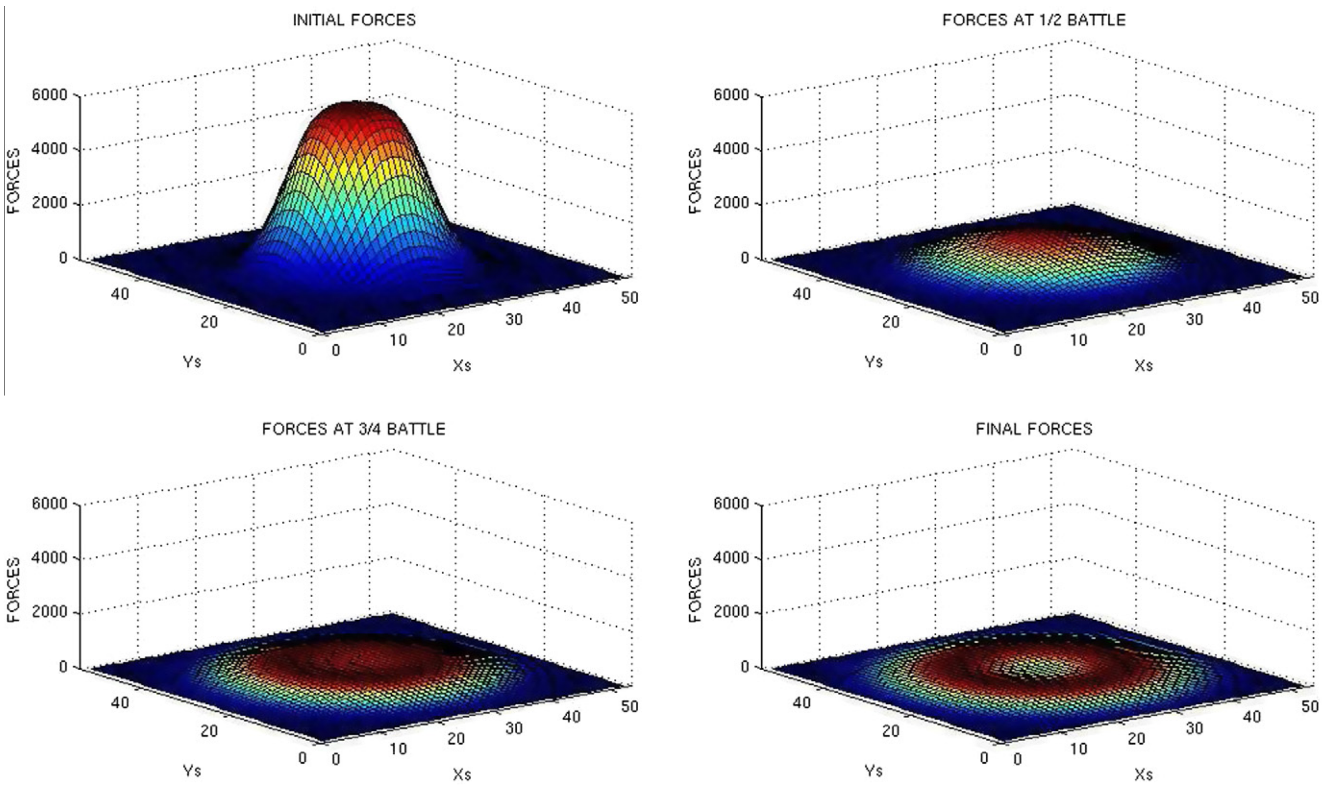

Fig. 5. Evolution of Red forces for centered combat. $R(x,y)$ vs. spatial coordinates.

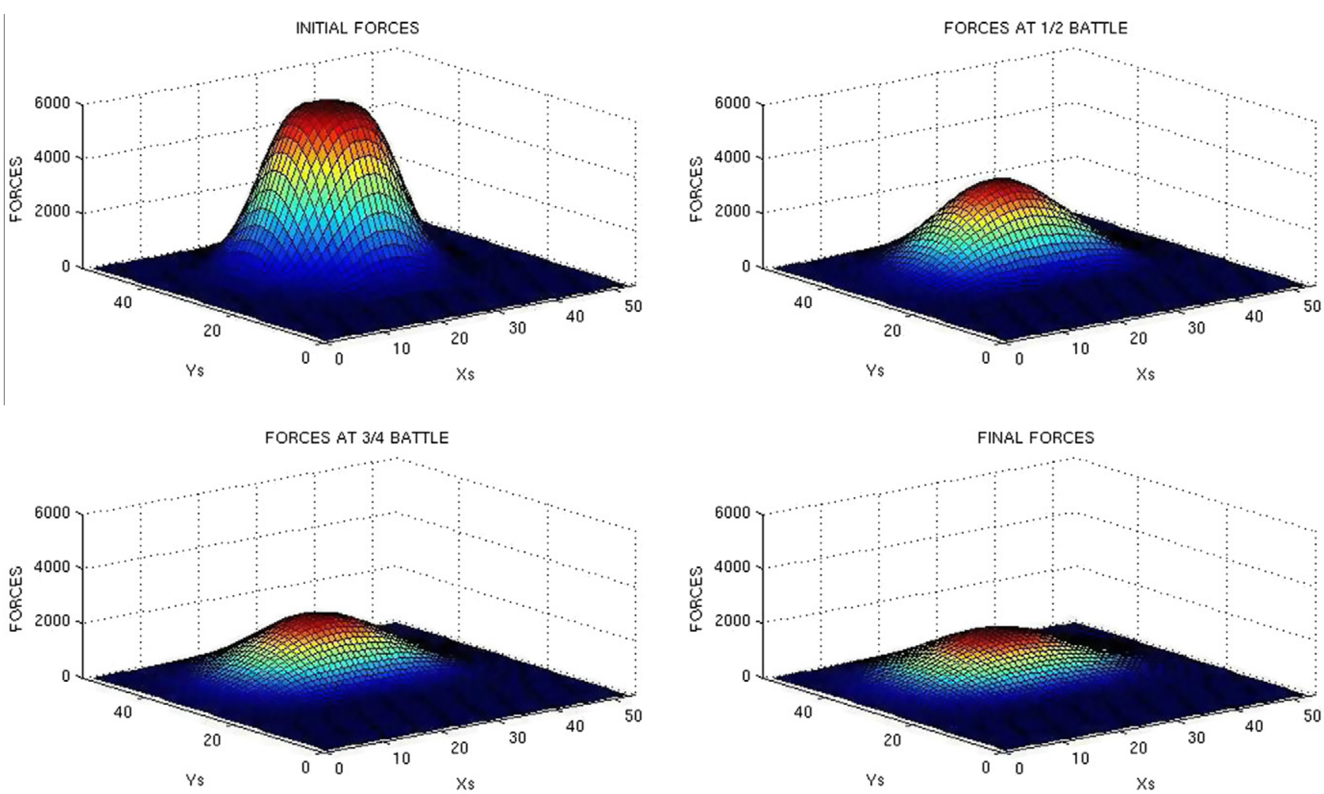

Fig. 6. Evolution of Red forces for shifting. $R(x,y)$ vs. spatial coordinates.

is obtained about the final location of the forces. Despite the Blue army having well trained troops, in both analyzed cases, the attempt of the Red forces to destroy that sieging army ends up in an successful rebellion, where it seems to be a better strategy for them to have the forces in contact but not sharing the same shape of the Blue forces distribution. In fact, the result of these combats

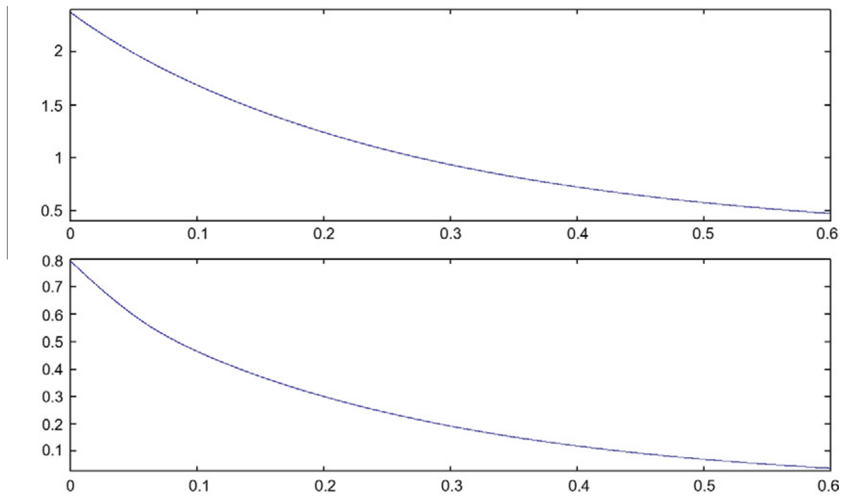

Fig. 7. Time evolution of the total forces for centered combat. Top: Blue vs. time. Bottom: Red vs. time.

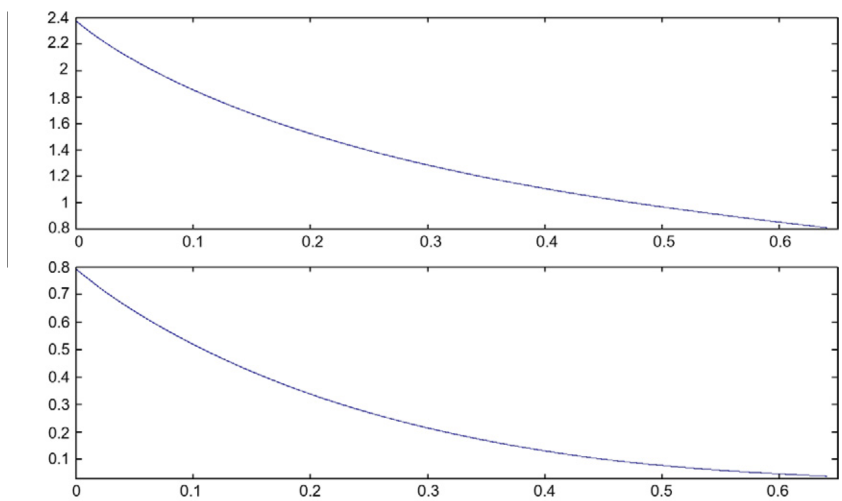

Fig. 8. Time evolution of the total forces for shifting. Top: Blue vs. time. Bottom: Red vs. time.

show for the centered combat that the Red forces finish with 4749 soldiers while for the off-centered combat the rebels end up with 8124 soldiers. (In both cases the Blue forces are defeated when they reach the size of 396 soldiers). An interesting situation can arise if there is a threat of a small reinforcement for the sieging army. If that is the case, maybe it would be wise to take the decision of rebelling for a shorter combat rather than a strategy of minimizing losses.

From the last two figures, Figs. 7 and 8, both shown using UT and US units, it is evident that the damage inflicted on the Blue forces is less significant if the rebels of the Red forces place their force distribution with a similar distribution to that exhibited by the Blue forces, even though the final result is plain defeat for the Blue forces. Classical Lanchester equations cannot predict the forces distribution once the combat has finished, Figs. 5 and 6, show this feature of the spatial modeling. Also an examination of the space–time evolution could be helpful in planning for eventual reinforcement of troops.

Another simulation that makes a reversal in the situation takes a change in amplitude (21,993 soldiers) and a slight relative spreading of the sieging army ($c_B = 20$), yielding 250 Red soldiers against 11,881 Blue soldiers in 920.2 minutes, with all the remaining rebels quartered, and still fighting, in the center of the battlefield.

More research could be done on trying to obtain, if possible, a distribution of the Blue forces that could lead to a reversal in the result of this combat for the same given distribution of the Blue forces.

7. Conclusions and outlook

A clear link between partial differential equations (Reaction–Diffusion equations) and Lanchester formulations is established, including the attitudes and perceptions of each force towards its opponents. The general model proposed here closes a gap not addressed in previous formulations, giving sense and interpretation to those previous models presented on the bases of analogies with other fields of knowledge.

Once the explicit movement dynamics and balance of forces are incorporated to the traditional LEs, this new Lanchester Spatial Model can be seen as a general formulation, where all the few attempts done in this direction in the literature are particular cases which could have a consistent comparison and review.

Furthermore, an original model of responsive movement is developed. As shown in the comparison to other formulations, before this work only one attempt to model velocity as a function of the opposite forces has been done. Here, troops are considered to move towards or away from the enemies depending on the balance of forces. This new feature allows to model local attrition in a more realistic way.

In this new formulation, it is possible to confirm that location influences the results of modeling attrition conflict between two opposite forces. The spatial distribution of the forces, their concentration and movement (diffusion) capabilities affect the overall results of the traditional LEs. The combat between an occupation army and a local army presented here can also be extended to a one-to-one conflict between special forces that infiltrate an enemy camp.

Specifically, it is shown that spatial concentration of forces will affect the time of annihilation and army's losses. Thus, the optimum location of forces needed to minimize cost or maximize damage is not intuitive and hence requires further investigation, most likely in the field of dynamic optimization (optimal control). However, armies' size, effectiveness and availability of supplies are still crucial to model the battle.

The model can also be extended, with suitable computational resources, quite straightly to more massive competing forces in the struggle. In the same way, resource partitioning in a military conflict could be easily incorporated. For instance, the work done by Sheeba and Ghose (2005) can be enhanced using our formulation.

An important conclusion of the model is that a new square law must be established when space is taken into account, since the disaggregated analysis of a battle can not be modeled on a spot, unless there exist both spatial homogeneity among the forces, and negligible movements of the struggling forces constrained to a small domain. If these two conditions are not met, the spatial Lanchester approach becomes a better tool for combat modeling while the square law does not survive as such. The same conclusion applies to the linear law.

Clearly, the results of the LEs are diverted by these new modeling possibilities in a number of directions, giving the modeler a whole new spectrum of variables and parameters to simulate in a more realistic fashion the current problems modeled with LEs. Additionally, new valuable information is generated by the model, since the final location and distribution of the armies can be determined, and not just the time of annihilation.

Given the general approach employed here, it is easy to adapt this model to allow different force movements and strategic behaviors to an even wider range of problems and applications, including 3D situations. Furthermore, the space–time modeling of antagonistic forces could improve a number of applications such as: crime prevention, marketing strategies, epidemiology, population evolution, pollution interaction, economic modeling, among many others.

Finally, this publication presents hints for further work, enhancing the search for solving attrition-like problems that benefit from the spatial dimension, as the introduction of remotely driven attraction/repulsion behavior of the forces, as shown in Subsection 3.4. This general model can have stochastic analysis, either by specifically modeling the velocity field or by a characterization of the term that accounts for the struggle.

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